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Consequential implications of mathematics student teachers' definitions of the function concept

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ABSTRACT

This article reports consequential implications of mathematics student teachers' definitions of the function concept. The implications emanated from scrutiny of written definitions, and exploration of demonstrated ability to identify functions and translate them into different representations. A qualitative study characterized by a case study design was conducted. Four student teachers of mathematics education at a public university constitute the sample. Whereas the study site was conveniently chosen, the participants were a sub-sample in the principal study selected using extreme case strategy. Data were collected through semi-structured interviews preceded by student teachers' written definitions of the function concept. Explorations of the written work and interview transcripts suggest that the student teachers' definitions of a function were dominated by a narrow view that all functions are one-to-one relations. Notwithstanding, the participants' conception of one-to-one functions was superficial. The student teachers' flawed definitions of a function influenced their inability to correctly identify functions. Likewise, those definitions were consistent with the student teachers' incapacity to translate functions accurately from one kind of representation into another. These findings underscore the necessity for mathematics teacher educators to facilitate student teachers' development of correct definitions and appropriate concept images of the function concept.

INTRODUCTION

Teaching for learners' conceptual understanding of the intended mathematics concepts should not only be an aspirational undertaking, but it ought to be cultivated. A factor that does contribute to successful learning of mathematics is a classroom environment in which learners actively 'do' mathematics (Van de Walle et al., 2013). However, school learners' conceptual understanding of mathematics concepts is, indisputably, not only dependent on such a classroom environment and activities orchestrated by teachers, but also on the teachers' accuracy when explaining concepts. This suggests that mathematics teachers' ability to comprehensively explain mathematics concepts and provide appropriate justifications for their reasoning are critical ingredients of teacher knowledge (Malambo, 2015). While it is inevitable that in countries like Zambia which champion social constructivism in the school curricula (Ministry of Education, 2013a) learners are expected to construct knowledge socially, a teacher should not be the worst facilitator in such an environment. This view is admissible because the explanations about mathematics concepts which teachers provide during lessons have an influence on the mathematical understandings acquired by learners.

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In this regard, implications of what mathematics teachers understand, say, and do about school mathematics concepts require attention in mathematics education research.

Van de Walle et al. (2013) posit that if teachers are to give high quality mathematics education, they ought, among other things, to "…. select meaningful instructional tasks and generalizable strategies that will enhance learning" (p. 3). This extract states the significance of the strategies, and methods including the selection thereof, which teachers employ when teaching mathematics. It should, however, be stated that teachers of mathematics are better to implement strategies that facilitate learning when they have in-depth understanding of the mathematics subject matter taught (Malambo, 2020; Malambo, 2021). In view of this, effort should be exerted to ensure that the understandings which teachers espouse about mathematics concepts are neither contradictory nor those that promote learners' development of misconceptions. Besides, as the National Council of Teachers of Mathematics (NCTM) (2000) recommends, teachers of mathematics should understand what learners understand and require for them to learn. By and large, that understanding should enable teachers to support learners to learn the intended material effectively.

Notwithstanding what I have written above concerning teachers of mathematics, it should be acknowledged that some of the characteristics of in-service teachers are likewise applicable to student teachers of mathematics. This is because when mathematics student teachers have graduated, they are likely to be recruited as teachers of mathematics. Arguably, what student teachers understand of mathematics concepts by the end of their training is as important as their demonstrated ability to explain the concepts (Malambo, 2021). Moreover, there should be consistency between what student teachers specify to understand of a mathematics concept and what they demonstrate to know of that concept. One of the key concepts in mathematics curricula is that of a function which is also considered as a unifying concept (Nyikahadzoyi, 2013; Watson & Harel, 2013).

Over the years, and in different countries, research studies have been conducted focusing on students' conception of the function concept (Bardini et al., 2014; Bayazit, 2011; Chesler, 2012; Even, 1993; Even & Tirosh, 1995; Hitt, 1998; Nyikahadzoyi, 2013, Schwarz, Dreyfus, & Bruckheimer, 1990; Sierpinska, 1992; Spyrou & Zagorianakos, 2010; Thompson, 1994). In some of these studies, participants have demonstrated shallow understanding of a function concept (Even, 1993; Even & Tirosh, 1995). Students have exhibited difficulties to differentiate functions from ordinary relations and first year undergraduate mathematics students have failed to provide an appropriate definition of a function (Bardini et al., 2014; Spyrou & Zagorianakos, 2010). Furthermore, student teachers have exhibited challenges to reason with and about mathematical definitions of functions (Chesler,2012). Teachers have also shown that they face challenges to maintain the characteristics of a function after changing representation of a function. Last, but not least, teachers have portrayed an inclination to a definition of a function connected to the rule of correspondence unlike one that involves the idea of a variable (Hitt, 1998). Although it may be possible to draw generic inferences, the primary focus of each cited studies was not on consequential implications of the respondents' definitions of the function concept. Moreover, these are mostly case studies for which the findings, even when relatable, cannot be generalized. Furthermore, despite the case studies which have been conducted in some countries around the function concept, the Zambian context has a dearth of such research studies. A review of mathematics education research literature suggests that a pioneer study in Zambia was only conducted less than 10 years ago by the author of the current article (Malambo, 2015). This reality is although the function concept is taught in Zambian secondary schools and universities.

In Zambia, when teaching the function concept, the emphasis includes distinguishing ordinary relations from a function (Ministry of Education, 2013b). School mathematics textbooks provide a popular definition of a function which states that a relation is a function if and only if each object in a domain is linked to a unique image in the range (Kalimukwa et al., 1995). Secondary school learners are introduced to different types of relations such as the many-to-many, one-to-many, many-to-one, and one-to-one (Ministry of Education, 2013b). Prominence is given to the property that distinguishes an ordinary relation from a function as well as to the difference between general functions and one-to-one functions. Gradually, learners do study representations of functions like formula, tables of values, and Cartesian graphs which are premised on linear and quadratic functions. In addition, and without essentially stressing the arbitrariness of functions, pictorial diagrams like arrow diagrams are taught. These aspects are progressively taught to mathematics student teachers in university though functions at this level of education are normally presented in formula and graphical form. In respect to the one-to-one function, the following algebraic definition is taught in university: If for a function f, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_{1,} a_2 \in D(f)$, it follows that the function f is a one-to-one function. The preceding review confirms that the definition of the function concept is a critical component in the Zambian mathematics curricula just like it is for several other countries of the world.

Mathematics definitions play a significant role in the process of introducing, describing, understanding, and communicating concepts (Chesler, 2012; Kemp & Vidakovic, 2021; Vinner, 1983; Zaslavsky & Shir, 2005). Besides, mathematics definitions could be helpful in carrying out cognitive tasks which are an inevitable part of mathematics (Vinner, 1983). Unquestionably then, definitions of mathematics concepts are a significant part of the process of teaching and learning mathematics. The issue of definitions is tackled in Vinner (1983) as he discusses the constructs of concept definition and concept image. These constructs relate to how people acquire understanding knowledge of mathematics concepts and what may constitute the understanding of concepts. A concept definition is perceived as "a verbal definition that accurately explains the concept in a non-circular way" (Vinner, 1983, p. 293). Tall and Vinner (1981) distinguished a personal concept definition from a formal concept definition. Formal concept definitions involve the uses of words when specifying concepts and such definitions are accepted by the mathematics community (Tall & Vinner, 1981). Contrasting concept definitions from concept images, Vinner (1983) explains that a concept image about a concept as held by a person is regarded as a "set of properties together with the mental picture" (p. 293). A concept image is conceptualized to be non-verbal and that it includes visual representations. To that extent, an image of a concept likely depends upon the person having it. Tall and Vinner (1981) discussed the idea of an evoked concept which was described as "the portion of the concept image which is activated at a particular time" (p. 152). Even though it is possible that concept definitions could lead to concept images, this is not automatic as concept images could equally lead to concept definitions (Vinner, 1983).

The foregoing issues are relevant to Zambia where there is a lack of context-specific studies that focus on mathematics student teachers' personal definitions of mathematics concepts. Furthermore, a demonstration of a void in our knowledge of the implications of Zambian mathematics student teachers' definitions of the function concept has been made. This article is a step to begin to address the void in our knowledge by providing answers to the following research questions: (1) How can mathematics student teachers' definitions of a function concept be described? (2) What consequential implications are evoked by mathematics student teachers' definitions of a function concept? Answers to these questions contribute additional contextual literature to the mathematics education research community concerning mathematics student teachers' understanding of the function concept. The benefits of acquiring understanding of the experiences of mathematics student teachers about common mathematics concepts in different settings cannot be overemphasized. This is against the background that the structures of teacher preparation and content emphasized in mathematics teacher education programs across countries do vary. In this regard, the current article informs and contributes to our understanding of the consequential implications of the Zambian mathematics student teachers' definitions of the function concept.

METHODS

This article is based on a qualitative study that used a case study design (Merriam, 2009; Nieuwenhuis, 2014a). Four student teachers, in the final year, studying mathematics education courses in an undergraduate mathematics education program at a leading Zambian public university were the sample. These students were a sub-sample from the principal study and were selected for this article through an extreme case strategy. The extreme case strategy is characterized by intentional selection of participants who exhibit extreme characteristics or provide radical information (Creswell, 2012). In this regard, the four mathematics student teachers were intentionally selected because when compared with other student teachers in the principal study they provided extreme information that was radically insightful. Final-year student teachers were involved as they had studied the mathematics education core courses that are designed for prospective secondary school teachers of mathematics at the study site. Individual and face-to-face semi-structured interviews were conducted with each of the student teachers. The interviews were preceded by administration of a paper and pencil diagnostic test which focused on functions. The data collection questions which are relevant to this article are reported in Figure 1.

In the principal study, the questions in Figure 1 were among those administered to the sample initially through the test instrument. Furthermore, these formed a basis for some of the specific interview questions posed. Definitions of a function, one-to-one function, identifications, justifications, and translation from the symbolic to the graphical representation of a function were explored both through the student teachers' written and verbalized text. Given that the questions in Figure 1 are based on the subject matter in which I'm knowledgeable, I first read and re-read several times the students' written work and transcripts. The intention was to establish any dominant common or dissimilar ideas in the responses of the student teachers. Concurrently, the students' work was qualitatively explored to determine consequential implications. Overall, the student teachers' written work and interview transcripts were analyzed using content analysis to establish the trend and emerging ideas. Use of data from two different sources such as the written test and interviews assisted to check the findings and consequently enhanced trustworthiness (Nieuwenhuis, 2014b).

Schoenfeld (2007) contends that "even the simplest observations or data gathering are conducted under the umbrella of either implicit or explicit theoretical assumptions, which shape the interpretations of the information that has been gathered" (p. 70). This extract is consistent with Schoenfeld's interpretation of Einstein's perspective that "there are no data without theory and there is no theory without data" (p.70). In keeping with these observations concerning theory, I adapted characteristics of Specialized Content Knowledge (SCK) of the Mathematical Knowledge for Teaching (MKT) framework (Ball et.al., 2008) and developed descriptors. SCK was defined by descriptors to be student teachers' capacity to: (1) write appropriate definitions of a function and one-to-one function; (2) explain the function and one-to-one function concepts (3) justify reasoning; (4) change representation accurately of functions to other forms of representation; and (5) preserve meaning between the written, and verbalized definitions of a function concept and other representations of a function (Malambo, 2015; Malambo, 2019). The next section provides the applicable findings.

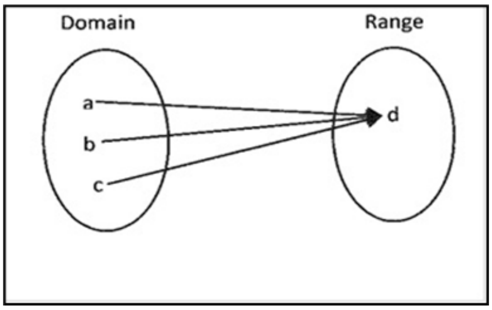
FINDINGS

The findings presented hereafter are in relation to questions in Figure 1 and the corresponding semi-structured interviews conducted. For each question, I will first give a brief explanation of what was being assessed followed by the results. Analyses of the students' written work and interview transcripts will be made concurrently. The results arising from items 1and 3 will be presented as they were generated through the test while those that hinge on items 2 and 4 are presented in the context of the interviews conducted. This is done to uphold only data relevant to the focus of this article. Results based on item 5 will include issues that arose from both the test and interviews. The definitions of a function and one-to-one function which were written by the student teachers will be provided in italics for easy identification. To enhance anonymity of the student teachers involved, I will use the following pseudonyms: Solomon, Joseph, Moses, and Daniel.

For question 1, it was expected of the participants to provide appropriate definitions of a function that encapsulated the sufficient characteristics of a function. In this regard, a definition was deemed appropriate if it was non-restrictive (not confining itself to regular rules and formulae) and highlighted the difference between an ordinary relation and a function. Thus, it was important for student teachers to demonstrate understanding that not every relation is a function and that for functions every object in a domain must be connected to a unique image in the range. In addition, student teachers required to demonstrate understanding that a function can be depicted diagrammatically without a definite rule or formula.

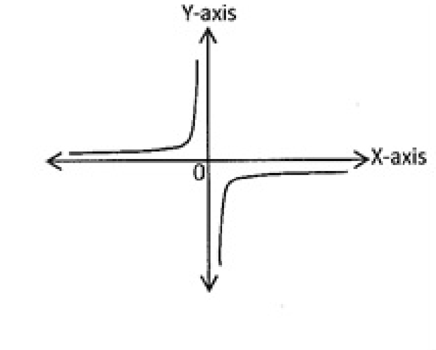
Daniel defined a function as '*a relation which is both one-to-one and many-to many*'. This view suggests that the student teacher lacked in-depth understanding of what a function is. While a one-to-one relation is a function, the student's declaration did not amount to a definition, but Daniel highlighted a mere name of a relation. Moreover, not all functions are one-to-one relations, and this fact makes the view of the participant shallow. The student teacher's statement that a function is also

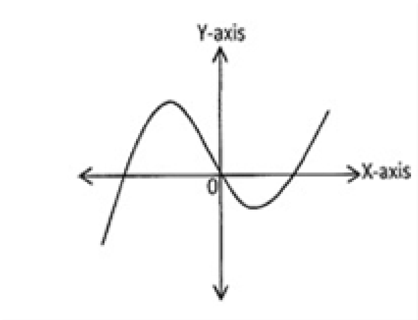
1. Give a definition of a function.
2. Indicate whether the figure below is a function or not and provide a justification.



Graph A

3. Write down a definition of a one-to-one function.
4. A mathematics textbook shows the following two graphs as examples of one-to-one functions. Indicate whether the textbook is correct in this regard, or not and provide justifications.





Graph B Graph C

5. Represent $g(x) = |x|$ whose domain is $\{x : -3 \le x \le 2 \text{ and } x \in Z\}$ on a Cartesian plane.

Figure 1. Items administered in the context of this article

a many-to-many relation provides evidence of lacking understanding of the property that qualifies a relation to be a function. The lack of understanding was also suggested by the student's assertion that for a relation to be a function, it should be both one-to-one and many-to-many. It is not possible practically to have a relation that is one-to-one and many-to-many at the same time. These considerations provide sufficient evidence that the student did not have a correct understanding of the definition of a function.

Joseph wrote that a function is '*a relation that maps one-to-one*'. This narrow and incorrect view that restricts a function to the aspect of one-to-one correspondence is consistent with the understanding articulated by Daniel. There was corroboration between the student teachers' written and verbalized definitions of a function as the following discussion with a participant named Solomon demonstrates:

I (Interviewer): Would you describe for me what a function is?
S (Solomon): A function, in my view, is a one-to-one relationship between sets. There must be one-to-one correspondence between elements of one set to the elements of the other.
I: In other words, you are saying that for a function there must be one-to-one correspondence?
S: One-to-one correspondence, yah [yes].
I: If there is no one-to-one correspondence then it means it's not a function?
S: It's not really a function.

Solomon's perspective excluded many-to-one relations from being functions. To acquire insight, the researcher cited an example of a many-to-one relation (an arrow diagram of the same was also presented to the respondent) and asked Solomon to confirm whether it is a many-to-one relation:

I: Suppose you have an illustration like this one where you have two sets. In set A you have elements 'a, b, [and] c', and in the next set B you have elements 'd' and 'e'. And you have 'a' related [linked] to 'd', 'b' related to 'd' and 'c' linked to 'e'. Would this be a many-to-one relation?
S: Mmm this one can't be a many-to-one [relation] because these three elements in set A are not all going to one element. It is only two elements that go to one element and this other element is going to another element, so you can say that (pauses), you can't say that this is specifically a many-to-one relation.

This narrow conception of functions demonstrates Solomon's lack of understanding of the correct definition of a function. It suggests that the respondent was not conversant with the properties of univalence and arbitrariness as they relate to functions.

Moses defined a function as a '*mapping of one element to the other by the use of a given rule*'. This definition portrayed a student teacher who held a restrictive view that functions are only those defined by rules which act on the referred elements. Furthermore, the definition did not bring out the property that distinguishes functions from ordinary relations.

Question 2 was assessing the student teachers' capacity to identify a function presented as an arrow diagram as well as their ability to provide appropriate justifications. In addition, the question formed the basis for investigation of the respondents' knowledge of the aspect that functions can be represented diagrammatically without stating formulas. The sampled student teachers could not identify the arrow diagram (Graph A) to be a depiction of a function. A discussion with a student teacher named Joseph highlights this:

I: Why would you take that view that Graph A is not a function?
J (Joseph): Like I said aah much as they [elements in the perceived domain] all have images, they have an image in the range. But they are mapped onto one image. There should be just one corresponding image for every object in the domain.

Joseph argued that Graph A is not a function because all objects are linked to one and the same image. By implication, the student teacher expected each object to have a different image. This view suggested that Joseph did not have in-depth knowledge of many-to-one relations and this discovery amplified the inaccuracy of Joseph's written definition of a function. Joseph provided the following inappropriate written definition of a function: A function is '*a relation that maps one-to-one*'. Joseph's failure to realize that for Graph A all objects in the domain were linked to a unique image in the range demonstrated a narrow conception of a fundamental condition that qualifies relations to be functions.

Question 3 was intended to assess the student teachers' ability to provide a definition that distinguishes a one-to-one function from a many-to-one function. It was required of the participants to demonstrate understanding that in a one-to-one function, all the objects in the domain have unique images in the range and that all images in the range also have unique objects in the domain.

A student teacher wrote that: '*a one-to-one function is where all the elements of the domain are mapped exactly to one point (or element) in the range. Also, all the elements of the range have exactly one object from the domain*'. This definition is lacking in clarity, for example, the student indicates that in a one-to-one function, 'all' objects have one image, but proceeds to state that 'all' images have one object. It is possible that the student may have been trying to indicate that a one-to-one function is such that each of its objects has a unique image and each of its images has a unique object. Thus, this could be a case of failing to write what one means, but regardless of that suspicion it suggests that the student teacher's definition of a one-to-one function was vague.

Another student teacher presented the following definition in relation to one-to-one functions: '*a one-to-one function is a function that maps each element in the domain to only one image in the range*'. The participant's definition suggests lack of understanding of the features that differentiate a one-to-one function from a many-to-one function. The student only attempted to bring out a feature that distinguishes an ordinary relation from a function. There was no effort made to highlight that the images of a one-to-one function have unique objects in the domain. This deficiency in the written definition is interesting especially that the student teachers' definitions of a function alluded to the one-to-one correspondence feature. The student teacher presented two arrow diagrams alongside the written definition. Those arrow diagrams are reproduced and denoted in Figures 2 and 3.

Figures 2 and 3 which are devoid of rules or formulae linking objects in perceived domains to elements in implied ranges suggest that the student teacher may have had an idea concerning the arbitrariness of functions. However, provision of pictorial representations without being prompted could suggest a deficiency in the student's capacity to provide an absolute word definition of a one-to-one function. Although Figures 2 and 3 represent functions in general, only Figure 2 is an accurate depiction of a one-to-one function. Presentation of a one-to-one function alongside a many-to-one function suggests that the student teacher was not sure of what a one-to-one function is. Interview findings corroborated the student teachers' inability to provide comprehensive written definitions as testified by the ensuing excerpts involving two student teachers:

> **M (Moses):** When you talk of a one-to-one function, there should be elements in the domain and elements in the range which are outputs. After substituting in the function, the given expression, so once aah I fuse in probably the one element in the expression, then the outcome should only be one. So, meaning that I'm not supposed to have more than two elements in the range. One element in the domain should map onto one element in the range or the output should only be one.
>
> **D (Daniel):** Like I have said already when you have two sets, we are saying a one-to-one function is a situation where you have one element being mapped onto one and only one element in the other set.

Moses confined one-to-one functions to formulae representations by stating that one must make substitutions in given expressions. Besides, the student attempted to define a one-to-one function using the univalence property and this suggests lack of understanding of the difference between many-to-one functions and one-to-one functions. Similarly, Daniel incoherently defined a one-to-one function using a property that distinguishes ordinary relations from functions. The following extract provides additional insight:

> **I:** Explain to me one difference that exists between a many-to-one function and a one-to-one function.
> **D:** A one-to-one function is a function where one element is being mapped onto one and only one [element] whereas a many-to-one, you have many elements being mapped onto one element in the range.

Daniel had no in-depth understanding of the difference between one-to-one and many-to-one functions and consequently demonstrated superficial understanding of the definition of a one-to-one function.

Question 4 provided the basis for investigating the student teachers' capacity to appraise whether each of the cartesian graphs represented a one-to-one function or not. Furthermore, the participants were required to provide justifications for positions espoused.

Solomon identified correctly Graph B of Figure 1 as a one-to-one function and supported this view by stating for $x > 0$ *and* $x < 0$ *every member of the x-axis is paired with exactly one member of the y-axis*. It should be noted that this explanation is only sufficient for a relation to qualify as a function. While it is a necessary condition in a one-to-one function, it is not sufficient. The student

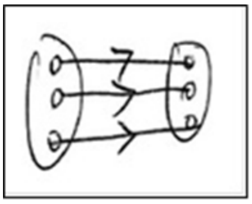


Figure 2.

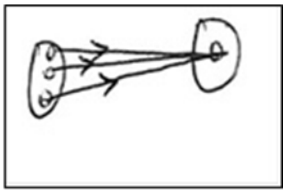


Figure 3.

teacher neglected to indicate that the *y*-values were also paired with unique *x*-values. Solomon's understanding of one-to-one functions was highly influenced by the held definition of a function in general. Solomon had indicated that a function is 'a one-to-one relationship between sets'. Another student teacher named Moses identified correctly Graph B as a representation of a one-to-one function, but he gave an inappropriate justification: '*Graph B is a one-to-one function since there is one value of y for every value of x*'. This view does not sufficiently speak to one-to-one functions as it does not absolutely exclude many-to-one functions. There seemed to be a way in which Moses' purported justification was influenced by an incorrect definition of one-to-one functions.

Moses identified correctly Graph C as not being a representation of a one-to-one function. When prompted to provide a justification, the student teacher stated that Graph C was not a depiction of a one-to-one function because a horizontal line cut Graph C at two points. This response was a mere description of the strategy employed to draw a conclusion and not a justification. A conflicted understanding manifested when Moses changed positions and contended that Graph C is a representation of a one-to-one function as the following excerpt confirms:

M: So now I notice that there is one *x* that can be paired to more than one *y* because when *x* is greater than zero I can obtain this point here, and when *x* is less than zero. Okay, I see, no I think I missed that one.
I: So you want to change; you are [now] saying it [Graph C] is not or it is an example of a one-to-one function?
M: This [Graph C] should be a one-to-one function.
I: A one-to-one function?
M: Yes.

The preceding extract does not only suggest Moses's inability to define a one-to-one function, but also provides evidence regarding how the held definition influenced failure to identify that Graph C is not a representation of a one-to-one function. The following is an excerpt from an interview with Daniel who posited incorrectly that Graph B is not a representation of a one-to-one function:

I: Why do you take that position that Graph B is not a [representation of a] one-to-one [function]?
D: Uh because aah one point is being mapped onto more than (pauses).
I: One point is being mapped onto more than what?
D: One element in the range.

Daniel's superficial definition of a one-to-one function seems to have influenced the failure to correctly identify that Graph B is a representation of a one-to-one function. Moreover, Daniel argued erroneously that Graph B could be a one-to-one function if '*each element in the domain is mapped onto one and only one element in the range*'. This perspective is only sufficient for relations to qualify as functions and is not sufficient although it is necessary for relations to be one-to-one functions. Daniel asserted that Graph C is a representation of a one-to-one function because '*one element is being* *mapped onto one and only one element'*. Somehow, a view which was used to disqualify Graph B from being deemed as a one-to-one function was used to assert incorrectly that Graph C is a representation of a one-to-one function. The incorrect identifications of Graphs B and C by Daniel were likewise made by Joseph except that the latter failed to provide justifications.

Question 5 was based on a function expressed in formula form and defined on a discrete domain (presented in set builder notation). The purpose was assessment of the student teachers' capacity to relate their definitions of a function to the task of changing representation of a function which is defined on a specified domain. Student teachers' ability to change representation to the graphical form while preserving the attributes of the symbolic function was likewise investigated. In this context, the question assessed student teachers' understanding of the implications of discrete and continuous domains as they relate to graphs of functions.

The four student teachers demonstrated ability to generate images of the function using the formula and some elements in the given domain. They then plotted the resultant ordered pairs on Cartesian planes. Figures 4 and 5 are representative of the graphs which were presented by the four student teachers.

Figure 4 shows seemingly accurate plotting on the Cartesian plane. Similarly, Figure 5 depicts visible points which are correctly plotted (though there is no visible table of values). Notwithstanding the minor differences and accuracy in plotting, both Figures 4 and 5 suggest that the student teachers who presented these figures lacked understanding of the nature of a graph of a function which is defined on a discrete domain. This view is derived from the act by the students of joining the plotted ordered pairs with straight lines. The ensuing excerpt from an interview with Moses is representative of the reasoning of the other student teachers in respect of Figures 4 and 5.

I: Do you have a reason why you connected the points?
M: Uh the reason why is because each time actually you are plotting, each time you are plotting, you have to connect the points.
I: Is that a rule?
M: Sometimes it is not always that you can (pauses); it has to be a straight line. So, depending on the Points plotted, it can either be a curve or a straight line.

Moses's claim that a graph of a function is either a curve or a straight line (without considering the nature of the domain) suggests a lack of in-depth understanding of the nature of graphs of functions that are defined on discrete domains. The written and professed definition of a function negatively impacted the ability of the student teacher to change representation of a symbolic function (formula form) to the Cartesian graph. Similar shallow conception of functions in several representations was manifest when Joseph considered $f(x) = x^2 + y$ to be aquadratic function because of the index 2 associated with the x-variable. During interviews, the student teachers were assessed regarding notation. They were specifically asked to indicate whether $y = 3x^2 + 4$ is a quadratic equation or not. The idea was to determine their perspective when y is used instead of $f(x)$. An interview scenario involving Joseph is reproduced hereafter:

I: Is $y = 3x^2 + 4$ a quadratic function?
J: This y is not telling us to say y is mapped onto this. Unless if you say y is mapped onto this, then that will make it to be a quadratic function.

The preceding excerpt is representative of the understandings demonstrated by all the four student teachers, and it suggests that the student teachers had difficulties with notation of functions. In the case of Joseph, this was confirmed when latter on the student indicated that $f(x) = 3x^2 + 4$ is a quadratic function simply because the y-variable was replaced by *f(x)*. Having presented the results, a discussion is provided in the following segment.

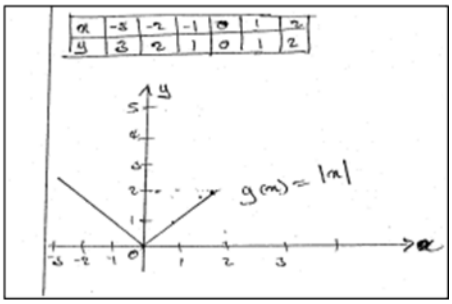


Figure 4.

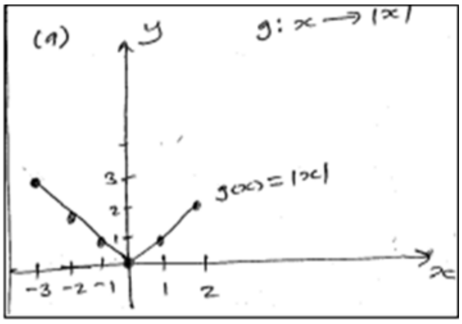


Figure 5.

DISCUSSION

The discussion hereafter is consistent with the essence of the descriptors highlighted in the last paragraph of the methods section. Mathematics student teachers' written definitions about a function suggest that the students lack capacity to define comprehensively a function. Their definitions depict superficial understanding which is dominated by the idea of one-to-one correspondence. In this regard, student teachers' definitions exclude many-to-one relations from being functions. There is a way in which the student teachers only think of one-to-one relations to be functions. This conclusion corroborates the findings of historical studies in which students thought that a relation is only a function when it upholds the one-to-one correspondence property (Leinharddt et al., 1990; Markovits, Eylon, & Bruckheimer, 1986).

One would have thought that since the student teachers referred to the one-to-one correspondence aspect when defining a function, they could demonstrate a correct written definition of a one-to-one function. Surprisingly, their definitions of one-to-one functions were aligned only with a flawed conception of the univalence condition. In other words, the student teachers attempted, incorrectly, to present a condition which is sufficient for relations to qualify as functions. They did so without realizing that such a condition although necessary was not sufficient when defining a one-to-one function. The univalence property acted as a didactical obstacle (Brousseau, 1997) to student teachers' comprehension of the concept of one-to-one functions. The finding alluded is consistent with the result of another study which suggests that students have difficulties to distinguish the idea of a one-to-one function from the property that qualifies relations to be functions (Dubinsky & Wilson, 2013).

Student teachers' deficient definitions of a function and one-to-one functions were also manifest through failure by the students to make correct identifications of functions in other representations. Evidently, a many-to-one function with a single image in the range and represented by an arrow diagram was considered as a non-function by the student teachers. Furthermore, the student teachers could not identify correctly one-to-one and non-one-to-one functions represented graphically. Under those circumstances, they attempted to use incorrect definitions to justify why they thought given graphs were either one-to-one functions or not. Likewise, they employed their incorrect definitions of a function to ascertain whether the arrow diagram (many-to-one relation) represented a function. The use of definitions of a function and one-to-one functions in this manner suggests that the student teachers' personal definitions influenced their capacity to identify functions and one-to-one functions.

Flawed definitions about a function also had an impact on the student teachers' capacity to accurately change representation of a function from formula to the Cartesian graphical form. The held incoherent definitions of the function concept, it seems, gave birth to inappropriate 'mental pictures' in student teachers concerning the Cartesian representation of a function. This claim could augment the view that concept images and concept definitions are interrelated (Vinner, 1983). In any case, the 'mental pictures' which the student teachers in the current study had of Cartesian graphs of functions were those defined on continuous domains. According to the student teachers, ordered pairs plotted in the Cartesian plane should always result in a curve or are supposed to be connected using straight lines. Apparently, the student teachers were able to remember 'certain properties of a function' but they could not articulate accurate definitions of the concept. This invigorated a claim that whenever a concept is introduced by definitions, the definitions do remain inactive or could get forgotten (Vinner, 1983). Besides, student teachers' inability to change a function representation accurately from a formula to a Cartesian graph suggested their inability to preserve meaning of a function in different representations.

An insightful scenario from the data suggests that student teachers may have 'mental pictures' suggesting that a written definition of a one-to-one function is only complete when complemented by pictorial representation. A student teacher, for example, gave two arrow diagrams (Figures 2 and 3) in addition to an incorrect written definition of a one-to-one function. Figure 2 portrays a one-to-one function while Figure 3 is a many-to-one function. Concurrent provision of Figures 2 and 3 as examples of a one-to-one function with a written definition intimated the student's conflicted understanding. Alternatively, depiction of unsolicited pictorial representations for a one-to-one function seems to lend credence to a claim that "in order to handle concepts one needs a concept image and not a concept definition" (Vinner, 1983, p. 293). Moreover, the student teacher exhibited superficial understanding in both the written definition and pictorial representations about a one-to-one function.

Another notable aspect worth discussing relates to a disconnect which was detected between the student teachers' professed definitions and formula representations of functions. This study has demonstrated that the student teachers' understanding of notation for function representation was characterized with the idea of a function acting on an object. They only accepted a formula to be a representation of a function when, for example, *f(x)* notation was utilized. This finding suggests that the student teachers had limited understanding which is consistent with the incorrect definitions of a function which they presented. This disclosure and the issues discussed earlier confirm that the student teachers in the current study could not 'flexibly and productively interact with mathematical definitions' (Chesler, 2012). It seems that there was a generic sense in which the student teachers in this study attempted to replicate a memorized definition of a function as depicted in Zambian school mathematics textbooks. An acknowledgement should be made that mathematics textbooks are critical as they play an influential role in deciding what and how of teaching (Van de Walle et al., 2013). However, it should be mentioned concurrently that research has shown that some mathematics textbooks lack appropriate definitions of mathematics concepts (Harel & Wilson, 2011). Anecdotal experience suggests that Zambian mathematics teachers rely on textbooks when preparing lesson plans. The mathematics textbooks scrutinized by the writer of this article have a generic pattern where definitions of concepts are first presented followed by examples (calculations) and exercises when dealing with topics. This pattern enshrined in textbooks is normally followed by teachers when teaching mathematics.

So far, it has been established that student teachers of mathematics do write and verbalize mathematical definitions of a function which in actual sense they do not imply. Thus, they may not always understand the mathematical definitions which they on face value seem to have ability to verbalize or write. In this regard, there is an inclination among student teachers to merely memorize the definition of a function and one-to-one functions without relational understanding (Skemp, 2006). Granted that one's 'mental pictures' concerning a concept do not necessarily arise from the definitions (Vinner, 1983), student teachers should, notwithstanding, develop a culture of using words correctly when defining a function. Fundamentally, they must always 'say' what they "mean" and "mean" what they 'say' about mathematics concepts. This is important owing to the possibility of definitions exerting influence either overtly or subtly on what is understood of mathematics concepts by learners. As tomorrow's teachers, student teachers should be trained to give particular attention to the mathematical 'language', symbols or terms used when defining a function concept. Symbolism in mathematics is considered as one of the ways of communicating mathematical ideas (Van de Walle et al., 2013). Mathematics definitions of concepts do conventionally use terms or symbols not in an exact manner such terms or symbols are used in everyday life. For example, in English the symbol '!' represents an exclamation mark while in mathematics it signifies a factorial.

CONCLUSION

This study has demonstrated that student teachers' definitions of a function concept do wield consequential implications. It has been shown that student teachers' definitions of a function concept tend to influence their ability to identify functions in general and one-to-one functions. The sampled student teachers' written, and verbalized definitions of a function tended to influence their justifications for positions held as they attempted to identify functions and non-functions. Furthermore, definitions provided demonstrated that the student teachers had a restrictive understanding of the function concept. This superficial understanding somehow limited student teachers' ability to understand functions expressed in different representations and yet ability to accurately change representation is an essential ingredient to enhancement of understanding of newly formed ideas (Van de Walle et al., 2013).

Since definitions of concepts do influence identifications of concepts in different representations, effort should be made to ensure that student teachers develop appropriate definitions and concept images of the function concept. Student teachers could be enabled to explore interrelationships between their personal definitions and the acquired concept images. Thus, as mathematics teacher educators teach the function concept, student teachers' concept images could be invoked concurrently. In the current study, the student teachers grappled with the use of phrases such as 'one image in the range', 'one and only one image', and 'unique image'. The student teachers were confusing these phrases with the construct 'one-to-one correspondence'. They were also confusing the univalence property with the phrase 'one-to-one correspondence' as it relates to one-to-one functions. The one-to-one correspondence aspect was an impediment to the student teachers' conception of the idea of 'unique image' for every object in the domain. Furthermore, the statement 'one image' for every object in a domain was interpreted as only one image and the same image' in the range for all objects in a domain. These revelations are an invitation to mathematics teacher educators to accord student teachers opportunities to explore these constructs in-depth. It is also important for trainee teachers to have chances for development of the capacity to avoid obstacles (Brousseau, 1997), and appreciation of the connectivity of mathematics concepts (Malambo, 2021).

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BIBLIOGRAPHY

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special? *Journal of teacher education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bayazit, I. (2011). Prospective Teachers' Inclination to Single Representation and Their Development of the Function Concept. *Educational Research and Reviews, 6*(5), 436-446.

Bardini, C., Pierce, R., Vincent, J., & King, D. (2014). Undergraduate Mathematics Students' Understanding of the Concept of Function, *IndoMS-JME,* Volume 5, No. 2, pp. 85-107. https://doi.org/10.22342/jme.5.2.1495.85-107

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer.

Chesler, J. (2012). Pre-service Secondary Mathematics Teachers Making Sense of Definitions of Functions. *Mathematics Teacher Education and Development, 14*(1), 27-40.

Creswell, J. W. (2012). *Educational Research Planning, Conducting, and Evaluating Quantitative and Qualitative Research* (Fourth ed.). United States of America: Pearson Education.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*(1), 83-101. https://doi.org/10.1016/j.jmathb.2012.12.001

Even, R. (1993). Subject-Matter Knowledge and Pedagogical Content Knowledge: Prospective Secondary Teachers and the Function Concept. *Journal for Research in Mathematics Education, 24*(2), 94-116. https://doi.org/10.2307/749215

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*(1), 1-20. https://doi.org/10.1007/BF01273897

Harel, G., & Wilson, S. (2011). The state of high school textbooks. *Notices of the AMS, 58*(6), 823-826.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *The Journal of Mathematical Behavior, 17*(1), 123-134. https://doi.org/10.1016/S0732-3123(99)80064-9

Kalimukwa, J. K., Mambwe, L., Mukuyamba, A., Mumbula, Y., Mwanakatwe, J. M., Nkhalamo, J., Shamapango, L., & Sichilima, J. (1995). *Mathematics Grade 10 Zambian Secondary School Syllabus*. Lusaka: Macmillan Publishers (Zambia) Ltd.

Kemp, A. & Vidakovic, D. (2021). Ways secondary mathematics teachers apply definitions in Taxicab geometry for a real-life situation: Midset. *Journal of Mathematical Behavior*, 62(2021), https://doi.org/10.1016/j.jmathb.2021.100848

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of educational research, 60*(1), 1-64. https://doi.org/10.3102/00346543060001001

Malambo, P. (2021). Implicit Misconceptions in Prospective Mathematics Teachers' Reasoning About Trigonometric Concepts. *Contemporary Mathematics and Science Education*, 2(2), ep21011. https://doi.org/10.30935/conmaths/11054

Malambo, P. (2020). Pre-service mathematics teachers' nature of understanding of the tangent function. *JRAMathEdu (Journal of Research and Advances in Mathematics Education)*, 5(2), 105-118. https://doi.org/10.23917/jramathedu.v5i2.10638

Malambo, P., Putten, S., Botha, H., & Stols, G. (2019). Dysfunctional functions: The case of Zambian mathematics education students. *Eurasia Journal of Mathematics, Science and Technology Education*, 15(1), em1651. https://doi.org/10.29333/ejmste/99510

Malambo, P. (2015). Exploring Zambian mathematics student teachers' content knowledge of functions and trigonometry for secondary schools (University of Pretoria). http://hdl.handle.net/2263/52943

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the learning of mathematics*, 18-28.

Merriam, S. B. (2009). *Qualitative Research A Guide to Design and Implementation* (Second ed.). San Francisco, CA 94103-1741: Jossey-Bass.

Ministry of Education. (2013a). *Zambia Education Curriculum Framework*. Lusaka: Zambia Curriculum Development Centre.

Ministry of Education. (2013b). *"O" Level Mathematics Syllabus (Grades 10 to 12).* Lusaka, Zambia: Zambia Curriculum Development Centre.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nieuwenhuis, J. (2014a). Qualitative research designs and data gathering techniques. In K. Maree (Ed.), *First steps in research* (14th impression ed., pp. 69-97). Van Schaik.

Nieuwenhuis, J. (2014b). Analysing qualitative data. In K. Maree (Ed.), *First Steps in Research* (Fourteenth impression ed., pp. 98-122). Pretoria: Van Schaik Publishers.

Nyikahadzoyi, M. R. (2013). Teachers' Knowledge of the concept of a function: A theoretical framework. *International Journal of Science and Mathematics Education*, 1-23.

Schoenfeld, H. (2007). Method. In K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Volume 1, pp. 69-107). Information Age Publishing Inc., United States of America.

Schwarz, B., Dreyfus, T., & Bruckheimer, M. (1990). A model of the function concept in a three-fold representation. *Computers & Education, 14*(3), 249-262. https://doi.org/10.1016/0360-1315(90)90008-U

Sierpinska, A. (1992). On understanding the notion of function. *The concept of function: Aspects of epistemology and pedagogy, 25*, 23-58.

Skemp, R. (2006). Relational Understanding and Instrumental Understanding. *Mathematics Teaching in the Middle School, 12*(2), 88-95. https://doi.org/10.5951/MTMS.12.2.0088

Spyrou, P., & Zagorianakos, A. (2010). Greek students' understandings of the distinction between function and relation. *Research in Mathematics Education, 12*(2), 163-164. https://doi.org/10.1080/14794802.2010.496988

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151–169. https://doi.org/10.1007/BF00305619

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. *Research in collegiate mathematics education, 1*, 21-44. https://doi.org/10.1090/cbmath/004/02

Van de Walle, J., Karp, K., & and Williams, J. (2013). Elementary and Middle School Mathematics: Teaching Developmentally, Pearson Education, Inc, United States of America.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14(3), 293-305. https://doi.org/10.1080/0020739830140305

Watson, A., & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with illustrations from two cases. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 154-168. https://doi.org/10.1080/14926156.2013.784826

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36(4), 317–346.